Available online at www.sciencedirect.com
SCIENCE DIRECT®
International Journal of Thermal Sciences 44 (2005) 245–254
International Journal of Thermal Sciences
www.elsevier.com/locate/ijts

A direct method to calculate the heat transfer coefficient of steady similar boundary layer flows induced by continuous moving surfaces

E. Magyari *, B. Keller

Chair of Physics of Buildings, Institute of Building Technology, Swiss Federal Institute of Technology (ETH) Zürich, CH-8093 Zürich, Switzerland

Received 19 April 2004; received in revised form 19 August 2004; accepted 19 August 2004

Abstract

The heat transfer coefficient mentioned in the title is calculated in this paper without to know the solution of the corresponding boundary value problem (neither in an explicit nor in an implicit form). This "direct method" makes use of a slightly modified form of the Merkin transformation which reverses the role of the dimensionless stream function f from that of an "old" dependent variable to one of a "new" independent variable and then gives the formal solution for the velocity and temperature profiles as a power series of $1 - f/f_\infty$ (where f_∞ denotes the dimensionless entrainment velocity). The series solutions yield for $f = 0$ explicit analytical expressions for the skin friction and the heat transfer coefficient of the flow. Owing to a formal mathematical analogy, the results also apply to the free convection boundary layer flows from vertical surfaces adjacent to fluid saturated porous media.

Keywords: Heat transfer; Boundary layer; Stretching surfaces; Similarity; Merkin transform

1. Introduction

The main goal of the present paper is to present a "direct method" which is suitable for an analytical calculation of the basic mechanical and thermal characteristics of the velocity and temperature boundary layers induced in quiescent viscous fluids by continuous stretching surfaces. Owing to a formal mathematical analogy, the results concerning the velocity boundary layer can be transferred to the case of free convection boundary layer flows from vertical surfaces adjacent to fluid saturated porous media.

The approach will be referred to hereafter as "direct method" since it does not require the knowledge of the solutions of the mentioned boundary value problems, neither in an explicit nor in an implicit form. It is based on a simple variable transformation which reverses the role of the dimensionless stream function f from that of an "old" dependent variable to one of a "new" independent variable. The idea of such a "role reversal" is quite old. In line with the general theory of pressure gradient driven steady boundary layer flows the stream function has been used as an independent variable already by von Mises [1] and in connection with the similar velocity boundary layers considered in the present paper by Merkin [2]. In the linear stability theory of solitary waves the use of an unknown base solution as an independent variable has also been proven to be an efficient tool (Magyari [3]). In the present paper we make use of a slightly modified form of the Merkin transformation [2] and transcribe Merkin's results [2] for the velocity boundary layer in a form which is instrumental for a rapid numerical evaluation. The analytical results for the thermal characteristics of the boundary layer flows induced by stretching surfaces represent the main contribution of the present paper.

Owing to their important industrial and geological applications, the two types of boundary layer flows mentioned above have attracted in the latter decades a vast research interest. The investigation of boundary layer flows induced by stretching surfaces has been initiated by the pioneering work of Sakiadis [4]. In the seminal paper of Banks [5] a compre-

* Corresponding author. Tel.: +01-6332867, fax: +01-6331041.
E-mail address: magyari@hbt.arch.ethz.ch (E. Magyari).

1290-0729/$ – see front matter © 2004 Elsevier SAS. All rights reserved.
doi:10.1016/j.ijthermalsci.2004.08.002

Nomenclature

A	dimensionless expansion coefficients, Eq. (29)
B	dimensionless expansion coefficients, Eq. (73)
f	dimensionless stream function, Eq. (7)
F	Gauss' hypergeometric function, Eq. (64)
G	dimensionless function, Eq. (73)
h	dimensionless heat transfer coefficient, Eq. (48)
k	thermal conductivity, Eq. (6b)
m	stretching exponent, Eqs. (9)
M	confluent hypergeometric function, Eq. (55)
n	temperature exponent, Eqs. (9)
Pr	Prandtl number, Eq. (16)
q	heat flux
S	skin friction, Eq. (25)
T	temperature, Eq. (1c)
u	dimensional longitudinal velocity, Eq. (11)
v	dimensional transversal velocity, Eq. (12)
x	dimensional wall coordinate
y	dimensional transversal coordinate
Y	dimensionless function, Eq. (22b)
z	dimensionless variable, Eq. (22a)

Greek symbols

α	thermal diffusivity, Eq. (1c)
β	stretching parameter, Eq. (20a)
γ	temperature parameter, Eq. (20b)
μ	dynamic viscosity, Eq. (6a)
ν	kinematic viscosity, $=\mu/\rho$
η	similarity variable, Eq. (7b)
ξ	dimensionless variable, Eq. (63)
ψ	dimensional steam function, Eq. (4a)
τ	shear stress
θ	dimensionless temperature, Eq. (7c)
Θ	dimensionless function, Eq. (45b)

Subscripts

w	wall conditions
∞	conditions at infinity
k	index, summation index
n	index, summation index

Abbreviations

BBL	backward boundary layer
FBL	forward boundary layer

hensive analytical and numerical investigation of the self-similar Sakiadis flows has been presented. For more recent developments in this field, the Refs. [6–10] can be consulted. Concerning the free convection boundary layer flows from vertical surfaces adjacent to fluid saturated porous media (to which the results of the present paper also apply), the first results have been reported by Cheng and Minkowycz [11]. For later developments and a rich list of references in this research field we recommend the recent monograph of Pop and Ingham [12].

2. Basic balance equations and boundary conditions

When the buoyancy forces may be neglected, the steady velocity and thermal boundary layers induced by a continuous impermeable stretching surface with temperature distribution $T_w = T_w(x)$ and stretching velocity $u_w = u_w(x)$ moving through a quiescent ($U_\infty = 0$) incompressible fluid of constant temperature T_∞ (Fig. 1) are governed in the boundary layer approximation by equations (see, e.g., [6–8])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1b}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{1c}$$

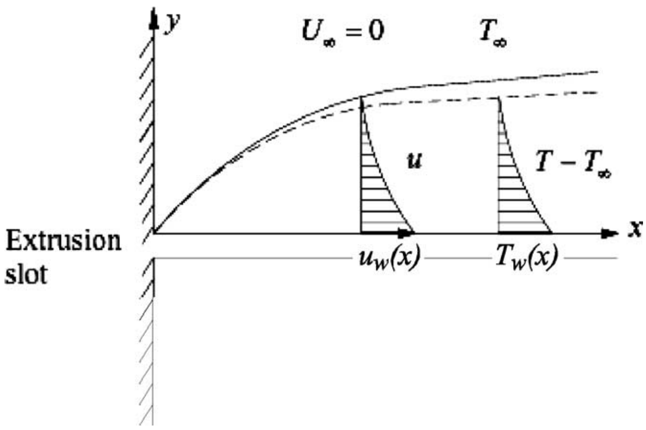

Fig. 1. Coordinate system, boundary conditions and steady forward boundary layers induced by a stretching wall issuing from a narrow linear slot.

along with the boundary conditions

$$u(x,0) = u_w(x) \tag{2a}$$

$$v(x,0) = 0 \tag{2b}$$

$$u(x,\infty) = 0 \tag{2c}$$

$$T(x,0) = T_w(x) \tag{3a}$$

$$T(x,\infty) = T_\infty = \text{const} \tag{3b}$$

The x-axis is directed along the continuous stretching surface and points from the narrow extrusion slot toward $+\infty$. The y-axis is perpendicular to x and to the direction of the slot (z-axis), u and v are the x and y components of the velocity field, respectively.

In the usual manufacturing situation (shown in Fig. 1) the surface issues from the slot and gives thus rise to a "forward boundary layer" flow (FBL) which moves from the slot toward $x = +\infty$. These are the Sakiadis-type boundary layer flows [4]. In the opposite case in which the continuous surface coming from $x = +\infty$ enters the slot, we are faced according to the nomenclature introduced by Goldstein [13] with the occurrence of a "backward boundary layer" flow (BBL) which moves toward the slot ($x = 0$). Such situations can be encountered in thermal treatment (e.g., annealing) of metallic sheets, wires, in the glass-fibre production, etc. A concrete example of a free convection BBL induced during the cooling of a vertically moving low-heat-resistance sheet has been given investigated by Kuiken [14]. The first results concerning the free convection BBL's over cold semi-infinite vertically upwards projecting surfaces adjacent to a fluid saturated porous media have recently been reported by Magyari and Keller [15].

The boundary value problem (1)–(3) incorporates an independent flow boundary value problem and a forced thermal convection problem. In terms of the stream function $\psi = \psi(x, y)$ defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$ the system of Eq. (1) reduces to

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = \nu\frac{\partial^3\psi}{\partial y^3} \tag{4a}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{4b}$$

and the boundary conditions (2) to

$$\frac{\partial\psi}{\partial y}(x, 0) = u_w(x) \tag{5a}$$

$$\frac{\partial\psi}{\partial x}(x, 0) = 0 \tag{5b}$$

$$\frac{\partial\psi}{\partial y}(x, \infty) = 0 \tag{5c}$$

The mechanical and thermal characteristics of main interest in this paper are the skin friction $\tau_w(x)$ and the wall heat flux $q_w(x)$, respectively:

$$\tau_w(x) = \mu\frac{\partial^2\psi}{\partial y^2}(x, 0) \tag{6a}$$

$$q_w(x) = -k\frac{\partial T}{\partial y}(x, 0) \tag{6b}$$

3. Similarity transformations

It is well known (see, e.g., Schlichting and Gersten [16]) that a transformation of the form

$$\psi(x, y) = A(x)f(\eta) \tag{7a}$$

$$\eta = B(x)y \tag{7b}$$

$$T(x, y) = T_\infty + C(x)\theta(\eta) \tag{7c}$$

where

$$A(x) \neq \text{const} \quad \text{and} \tag{8a}$$

$$B(x) > 0 \tag{8b}$$

leads to two basic types of similarity solutions of the problem (4), (5), (3). These are the similarity solutions of power law type corresponding to

$$A(x) = A_0 x^{(m+1)/2} \tag{9a}$$

$$B(x) = B_0 x^{(m-1)/2}, \quad m \neq -1 \tag{9b}$$

$$C(x) = C_0 x^n \tag{9c}$$

and the similarity solutions of exponential type corresponding to

$$A(x) = Ae^{ax} \tag{10a}$$

$$B(x) = B_0 e^{ax} \tag{10b}$$

$$C(x) = C_0 e^{cx} \tag{10c}$$

(A_0, B_0, C_0, n, m, a and c are real constants).

The assumption $B(x) > 0$ which implies $\eta \geqslant 0$ can be adopted without any further restriction of generality. The assumption $A(x) \neq$ const on the other hand, has a basic significance. Indeed, as shown recently by Magyari et al. [17], in the case $A(x) =$ const (which for the power law similarity (9) means $m = -1$) the transformation (7a) is much too restrictive and the corresponding boundary value problem does not admit FBL solutions (neither for impermeable nor for permeable surfaces). However, if the surface is permeable and a suitable lateral suction of the fluid is admitted, this "missing" boundary layer solution can readily be found by a slight extension of transformation (7a), [17]. In the present paper we assume $m > -1$ in the case of power-law similarity (9) and $a > 0$ in the case of exponential similarity (10), respectively (see also below).

In both of the above similarity cases the components of the corresponding velocity fields are obtained as:

$$u(x, y) = u_w(x)f'(\eta) \tag{11a}$$

$$u_w(x) = A(x)B(x) \tag{11b}$$

$$v(x, y) = -A'(x)\left[f(\eta) + \frac{A(x)}{A'(x)}\frac{B'(x)}{B(x)}\eta f'(\eta)\right] \tag{12}$$

(The primes denote in the above equations derivatives with respect to the corresponding argument.)

According to Eq. (12) the (dimensional) entrainment velocity of the flow is given by

$$v(x, \infty) = -A'(x)f_\infty \tag{13}$$

where

$$f_\infty = \lim_{\eta\to\infty} f(\eta) \tag{14}$$

denotes the dimensionless entrainment velocity. It is assumed that f_∞ is finite and non-vanishing.

In the following it will be assumed that the stretching velocity $u_w(x)$ does not change sign for $x \geqslant 0$. Thus

$\operatorname{sgn} u_w(x) = +1$ (i.e., $A_0 > 0$) corresponds to FBL's (the continuous surface issues from the extrusion slot) and $\operatorname{sgn} u_w(x) = -1$ (i.e., $A_0 < 0$) corresponds to BBL's (the continuous surface enters the slot), respectively. In the present paper we consider only the case of FBL's ($\operatorname{sgn} u_w(x) = +1$) which are of the main practical interest. Thus by choosing in the power-law and the exponential case

$$\frac{2\nu B_0}{(m+1)A_0} = +1 \quad \text{and} \tag{15a}$$

$$\frac{\nu B_0}{a A_0} = +1 \tag{15b}$$

respectively (where $m > -1$ and $a > 0$), the boundary value problem (4), (5), (3) can be reduced to the ordinary differential equations

$$f''' + ff'' - \beta f'^2 = 0 \tag{16}$$

$$\frac{1}{Pr}\theta'' + f\theta' - \gamma f'\theta = 0 \tag{17}$$

along with the boundary conditions

$$f(0) = 0 \tag{18a}$$

$$f(0) = 1 \tag{18b}$$

$$f'(\infty) = 0 \tag{18c}$$

$$\theta(0) = 1 \tag{19a}$$

$$\theta(\infty) = 0 \tag{19b}$$

where

$$\beta = \frac{2m}{m+1} \tag{20a}$$

$$\gamma = \frac{2n}{m+1} \tag{20b}$$

$$m > -1 \tag{20c}$$

for the power-law similarity and

$$\beta = 2 \tag{21a}$$

$$\gamma = \frac{c}{a} \tag{21b}$$

$$a > 0 \tag{21c}$$

for the exponential similarity, respectively. As usual, $Pr = \nu/\alpha$ denotes the Prandtl number.

The boundary value problem (16)–(19) incorporates an independent flow boundary value problem (16), (18) and a forced thermal convection problem (17), (19), respectively. The former one has comprehensively been investigated in a seminal paper by Banks [5] for $-\infty < \beta + \infty$ both by analytical and numerical methods. In the range $-2 < \beta \leqslant +2$ (i.e., $-1/2 < m \leqslant +\infty$) the solutions described by Banks [5] correspond to the usual forward boundary layers (FBL's). At $\beta = -2$ (i.e., $m = -1/2$) the solution becomes singular and for $-\infty < \beta < -2$ (i.e., $-1 < m < -1/2$) no solutions exist. Banks [5] also gives numerical solutions for β-values in the range $\beta > +2$ (i.e., $m < -1$). It should be underlined, however, that these solutions do not correspond to forward but to backward boundary layers. This can easily be seen from Eq. (15a) which for $m < -1$ implies $A_0 < 0$, i.e., $\operatorname{sgn} u_w(x) = -1$. It is also worth mentioning here that in the physical context of free convection flows in saturated porous media Ingham and Brown [18] have proved rigorously that the problem (16), (18) does not admit FBL solutions in the whole range $m < -1/2$, i.e., neither for $-\infty < \beta < -2$ (i.e., $-1 < m < -1/2$) nor for $\beta > +2$ (i.e., $m < -1$).

The thermal boundary value problem (17), (19) has also been investigated by standard methods by several authors both for permeable and impermeable surfaces and for different stretching velocities and surface temperature distributions (see Section 5 below).

4. The velocity boundary layer

4.1. The Merkin transformation

In contrast to the standard approach (see, e.g., Banks [5]) the flow boundary value problem (16), (18) has been solved by Merkin [2] by subjecting it first to an inventive variable transformation. Merkin's transformation [2] reverses role of f from that of the old dependent variable to that of a new independent variable $\phi \equiv f_\infty - f$ and at the same time, it transfers the role of the dependent variable from f to $p(\phi) \equiv \mathrm{d}f/\mathrm{d}\eta$. For later convenience we modify the Merkin transformation slightly by changing to a new independent variable z and to a new dependent one $Y = Y(z)$, which we define as follows:

$$z = 1 - \frac{f}{f_\infty} \tag{22a}$$

$$Y = \frac{1}{f_\infty^2}\frac{\mathrm{d}f}{\mathrm{d}\eta} \tag{22b}$$

Then the boundary value problem (16), (18) becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(Y\frac{\mathrm{d}Y}{\mathrm{d}z}\right) + (z-1)\frac{\mathrm{d}Y}{\mathrm{d}z} - \beta Y = 0 \tag{23}$$

$$Y|_{z=0} = 0 \tag{24a}$$

$$Y|_{z=1} = f_\infty^{-2}, \quad 0 \leqslant z \leqslant 1 \tag{24b}$$

(The boundary condition $Y|_{z=0} = 0$ has been obtained from $f'(\infty) = 0$ and $Y|_{z=1} = f_\infty^{-2}$ from $f(0) = 0$, $f'(0) = 1$, respectively.)

The dimensionless skin friction $f''(0) \equiv S$ is obtained in this approach as

$$S = -f_\infty \frac{\mathrm{d}Y}{\mathrm{d}z}\bigg|_{z=1} \tag{25}$$

After the solution $Y = Y(z)$ of the boundary value problem (23), (24) has been found, the solution $f = f(\eta)$ of the original problem (16), (18) can be obtained in the implicit form $\eta = \eta(f)$ by quadratures,

$$\eta = -\frac{1}{f_\infty} \int_1^{1-f/f_\infty} \frac{\mathrm{d}z}{Y(z)} \tag{26}$$

For the skin friction S further two useful integral relationships can be deduced by standard operations. They are

$$S = -(1+\beta) f_\infty^3 \int_0^1 Y(z)\,\mathrm{d}z \tag{27}$$

and

$$S = -\frac{1+\beta}{2 f_\infty (2+\beta)} - (1+\beta) f_\infty^3 \int_0^1 zY(z)\,\mathrm{d}z \tag{28}$$

respectively.

4.2. The series solution

Looking for the solution of the boundary value problem (23), (24) in the form

$$Y = \sum_{n=0}^{\infty} A_n z^n \tag{29}$$

one obtains for the coefficients A_n the system of equations

$$\sum_{n=0}^{k} (n+1)\big[(n+2)A_{n+2}A_{k-n} + (k-n+1)A_{n+1}A_{k-n+1}\big] = (k+1)A_{k+1} + (\beta - k)A_k, \quad k = 0, 1, 2, \ldots \tag{30}$$

The boundary condition $Y|_{z=0} = 0$ implies immediately

$$A_0 = 0 \tag{31}$$

Thus, one obtains from (30) for the next two coefficients the expressions

$$A_1 = 1 \tag{32a}$$

$$A_2 = \frac{1}{4}(\beta - 1) \tag{32b}$$

The subsequent coefficients $A_3, A_4, A_5, \ldots$ can then be obtained recursively according to

$$A_k = \frac{\beta - k + 1}{k^2} A_{k-1} - \frac{k+1}{2k} \sum_{n=2}^{k-1} A_n A_{k-n+1}, \quad k = 3, 4, 5, \ldots \tag{33}$$

Specifically,

$$A_3 = \frac{1}{72}\big(1-\beta^2\big)$$

$$A_4 = \frac{1}{576}\big(1-\beta^2\big)(1-2\beta)$$

$$A_5 = \frac{1}{86400}\big(1-\beta^2\big)\big(11 - 81\beta + 88\beta^2\big)$$

$$A_6 = \frac{1}{1036800}\big(1-\beta^2\big)\big(-9 - 125\beta + 447\beta^2 - 337\beta^3\big) \tag{34}$$

Then the boundary condition (24b) yields for f_∞ the explicit expression

$$f_\infty = \left(\sum_{k=1}^{\infty} A_k\right)^{-1/2} \tag{35}$$

For the skin friction S, from Eqs. (25), (27) and (28) the following three (equivalent) expressions result:

$$S = -f_\infty \sum_{k=1}^{\infty} k A_k \tag{36}$$

$$S = -(1+\beta) f_\infty^3 \sum_{k=1}^{\infty} \frac{A_k}{k+1} \tag{37}$$

$$S = -\frac{1+\beta}{2(2+\beta) f_\infty} - (1+\beta) f_\infty^3 \sum_{k=1}^{\infty} \frac{A_k}{k+2} \tag{38}$$

The advantages of the modified Merkin transformation (22) consist of the following. By its use, both the differential equation (23) as well as the coefficients A_n of the series solution become independent of the dimensionless entrainment velocity f_∞. Thus, the expression (35) of f_∞ results from the boundary condition (24b) and the formal series solution (29) directly. In addition, the recurrence equation (33) allows for a rapid numerical evaluation of the coefficients A_n for any given value of β. In general, the absolute values of the coefficients A_k decreases with increasing k for any $-2 < \beta \leqslant 2$ quite rapidly, except for the neighborhood of the singularity at $\beta = -2$. For instance, the terms of the sequence $|A_k|$ become smaller than 10^{-6} already after its 8th term for $\beta = 0$ (remember that $A_1 = 1$ for any β), after its 20th term for $\beta = 2$, after its 27th term for $\beta = -1.5$, but for $\beta = -1.9$ they do only so after the 110th term of the series (29). Concerning the three (equivalent) expressions (36)–(38) of the skin friction S, the series (38) is (on obvious reasons) the most rapidly converging one.

4.3. Two cases of closed form solutions: $\beta = +1$ and $\beta = -1$

The case $\beta = +1$ ($m = +1$). In this case we obtain immediately $A_1 = 1$, $A_k = 0$, $k = 2, 3, 4, \ldots$. Thus,

$$Y(z) = z = 1 - f \tag{39a}$$

$$f_\infty = 1 \tag{39b}$$

$$S = -1 \tag{39c}$$

The corresponding implicit form of the solution results from Eq. (26),

$$\eta = -\int_1^{1-f} \frac{\mathrm{d}z}{z} = -\ln(z) = -\ln(1-f) \tag{40}$$

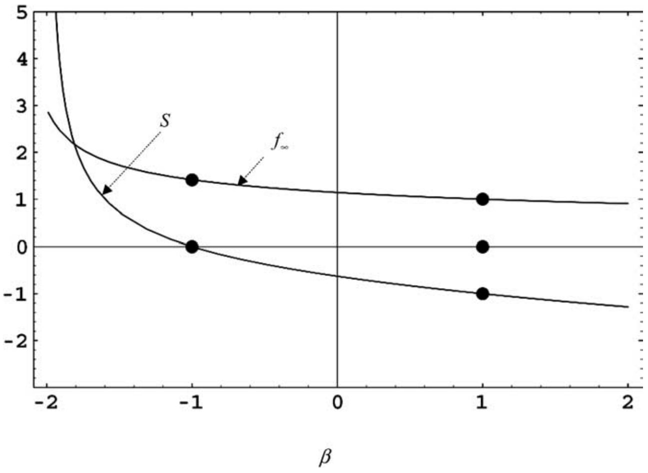

Fig. 2. Plot of the skin friction $f''(0) = S$ and of the dimensionless entrainment velocity f_∞ as functions of the parameter β, $-2 < \beta \leqslant +2$. The dots indicate the closed form solutions corresponding to the values $(\beta, S, f_\infty) = (-1, 0, \sqrt{2})$ and $(\beta, S, f_\infty) = (1, -1, 1)$. As β approaches the value -2 (i.e., as $m \to -1/2$) both S and f_∞ become singular. At $\beta = -1.808$ the two curves intersect each other and $f_\infty = S = +2.206$.

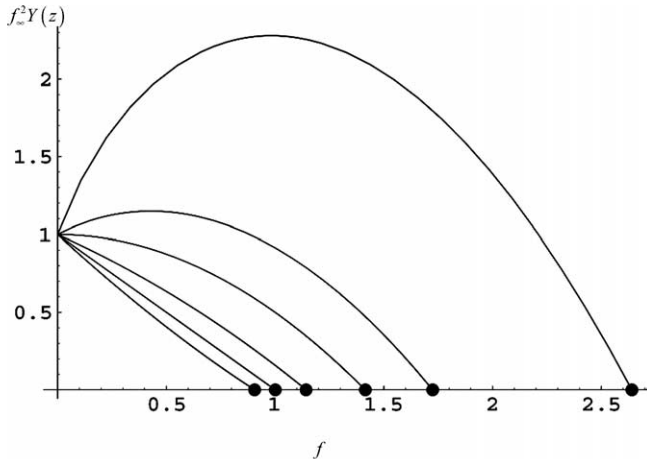

Fig. 3. Plot of the dimensionless velocity profiles $f'(\eta) = f_\infty^2 Y(z)$ as functions of the dimensionless stream function f for different values of the parameter β. The dots indicate the corresponding values of the dimensionless entrainment velocity f_∞. The β values, from top to the bottom, are $\beta = -1.9, -1.5, -1, 0, 1$ and 2, respectively. The corresponding values of the skin friction S and of the dimensionless entrainment velocity f_∞ are: $S = 3.92738, 0.72728, 0, -0.62758, -1$ and -1.28181, and $f_\infty = 2.6403, 1.7239, \sqrt{2}, 1.1427, 1$ and 0.9056, respectively.

The explicit solution reads

$$f(\eta) = 1 - e^{-\eta} \tag{41a}$$

$$f'(\eta) = e^{-\eta} \tag{41b}$$

$$f''(0) \equiv S = -1 \tag{41c}$$

We recover in this case the solution first found by Crane [19]. The value $S = -1$ of the skin friction can also immediately be recovered from Eqs. (36)–(38).

The case $\beta = -1$ ($m = -1/3$). We obtain in this case immediately $A_1 = 1$, $A_2 = -1/2$, $A_k = 0$, $k = 3, 4, 5, \ldots$. Thus,

$$Y(z) = z - \frac{1}{2}z^2 \tag{42a}$$

$$f_\infty = \sqrt{2} \tag{42b}$$

$$S = 0 \tag{42c}$$

The corresponding implicit form of the solution results from Eq. (26),

$$\eta = -\int_1^{1-\frac{f}{\sqrt{2}}} \frac{\mathrm{d}z}{Y} = \int_0^f \frac{\mathrm{d}f}{1 - f^2/2} = \sqrt{2}\,\mathrm{arc\,tanh}\left(\frac{f}{\sqrt{2}}\right) \tag{43}$$

The explicit solution reads

$$f(\eta) = \sqrt{2}\tanh\left(\frac{\eta}{\sqrt{2}}\right) \tag{44a}$$

$$f'(\eta) = \frac{1}{\cosh^2(\eta/\sqrt{2})} \tag{44b}$$

In Eq. (44) we recover the solution first written down by Bickley [20] in connection with the free plane jet (Schlichting–Bickley jet, [20,21]). The value $S = 0$ of the skin friction can also immediately be recovered from Eqs. (36)–(38).

As an illustration in Fig. 2 the skin friction $f''(0) = S$ and of the dimensionless entrainment velocity f_∞ are plotted as functions of the parameter β for $-2 < \beta \leqslant +2$. In the ranges $\beta < -2$ and $\beta > +2$ no FBL solutions exist (for $\beta < -2$, f_∞ becomes imaginary and for $\beta > +2$ the series (29) becomes divergent). In Fig. 3 the dimensionless velocity profiles $f'(\eta) = f_\infty^2 Y(z)$ are plotted as functions of the dimensionless stream function f for different values of β in the whole domain of existence $-2 < \beta \leqslant +2$ of the FBL solutions.

5. The temperature boundary layer

5.1. Transformation of variables

In the case of the forced thermal convection problem (17), (19) we perform the variable transformations

$$z = 1 - \frac{f}{f_\infty} \tag{45a}$$

$$\Theta = \frac{1}{f_\infty^2}\theta \tag{45b}$$

Here z is the new independent and $\Theta = \Theta(z)$ the new dependent variable, respectively. In this way Eq. (17) and the boundary conditions (19) become

$$\frac{1}{Pr}\frac{\mathrm{d}}{\mathrm{d}z}\left(Y\frac{\mathrm{d}\Theta}{\mathrm{d}z}\right) + (z-1)\frac{\mathrm{d}\Theta}{\mathrm{d}z} - \gamma\Theta = 0 \tag{46}$$

$$\Theta|_{z=0} = 0 \tag{47a}$$

$$\Theta|_{z=1} = f_\infty^{-2} \tag{47b}$$

(The boundary condition $\Theta|_{z=0} = 0$ has been obtained from $\theta(\infty) = 0$ and $\Theta|_{z=1} = f_\infty^{-2}$ from $\theta(0) = 1$, respectively.)

The dimensionless heat transfer coefficient $-\theta'(0) \equiv h$ is obtained in terms of the new variables as

$$h = f_\infty \frac{\mathrm{d}\Theta}{\mathrm{d}z}\bigg|_{z=1} \tag{48}$$

The integral relationship

$$h = (1+\gamma) Pr f_\infty^3 \int_0^1 \Theta(z)\,\mathrm{d}z \tag{49}$$

can also easily be deduced from Eqs. (46) and (47).

5.2. The Reynolds analogy for $\{\gamma = \beta,\ Pr = 1\}$

For $\gamma = \beta$, $Pr = 1$ and $\Theta(z) = Y(z)$ the thermal boundary value problem (46), (47) and the flow problem (23), (24) become identical (Reynolds analogy of the momentum and energy equations). Hence, we obtain in this case

$$\theta = f_\infty^2 Y(z) = \frac{\sum_{k=1}^\infty A_k z^k}{\sum_{k=1}^\infty A_k} = \frac{\sum_{k=1}^\infty A_k (1 - f/f_\infty)^k}{\sum_{k=1}^\infty A_k} \tag{50}$$

Similarly, the heat transfer coefficient h coincides with $-S$. Thus, we obtain also for h three equivalent expressions in this case,

$$h = f_\infty \sum_{k=1}^\infty k A_k \tag{51}$$

$$h = (1+\beta) f_\infty^3 \sum_{k=1}^\infty \frac{A_k}{k+1} \tag{52}$$

$$h = \frac{1+\beta}{2(2+\beta) f_\infty} + (1+\beta) f_\infty^3 \sum_{k=1}^\infty \frac{A_k}{k+2} \tag{53}$$

5.3. The case $\beta = +1$

In this case $Y = z$, $f_\infty = 1$ and $S = -1$ (see Eq. (39)). By the simple variable transformation

$$z = -\frac{\xi}{Pr} \tag{54}$$

the energy equation (46) goes over in

$$\xi \frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2} + (1 - Pr - \xi)\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} + \gamma \Theta = 0 \tag{55}$$

which is precisely Kummer's equation of the confluent hypergeometric functions $M(a, b, \xi)$ and $U(a, b, \xi)$ with $a = -\gamma$ and $b = 1 - Pr$ (see Abramowitz and Stegun [22]). Thus, the solution which satisfies the boundary conditions (47) can be put in the form

$$\Theta = z^{Pr} e^{Pr(1-z)} \frac{M(1+\gamma, 1+Pr, Pr\,z)}{M(1+\gamma, 1+Pr, Pr)} \tag{56}$$

The heat transfer coefficient (48) becomes

$$h = \frac{(1+\gamma)Pr}{1+Pr} \frac{M(2+\gamma, 2+Pr, Pr)}{M(1+\gamma, 1+Pr, Pr)} \tag{57}$$

Having in mind that in this case, according to Eqs. (39) and (41), $z = Y = e^{-\eta}$, we recover in (56) and (57) the results obtained by Grubka and Bobba [23] by standard calculations (see also [6,10]).

In order to be more specific we list below several special cases of Eqs. (56) and (57).

The special case $\gamma = Pr$. Having in mind that $M(a, a, x) = e^x$ we obtain from Eqs. (56) and (57)

$$\Theta = z^{Pr} = Y^{Pr} \tag{58a}$$

$$h = Pr \tag{58b}$$

The integral formula (49) gives obviously the same result

$$h = (1+Pr)Pr \int_0^1 z^{Pr}\,\mathrm{d}z = Pr \tag{59}$$

For $Pr = 1$ we recover the Reynolds analogy, $\Theta = Y$, $h = 1$.

The special case $\gamma = 0$, $Pr = 1$. Having in mind that $M(1, 2, x) = (e^x - 1)/x$, we obtain:

$$\Theta = \frac{e}{e-1}\left(1 - e^{-z}\right) \tag{60a}$$

$$h = \frac{1}{e-1} \tag{60b}$$

The special case $\gamma = -1$. Having in mind that $M(0, b, x) = 1$, we obtain:

$$\Theta = z^{Pr} e^{Pr(1-z)} \tag{61a}$$

$$h = 0 \tag{61b}$$

The special case $\gamma = -2$. Having in mind that $M(-1, b, x) = 1 - x/b$, we obtain:

$$\Theta = \left[1 + Pr(1-z)\right] z^{Pr} e^{Pr(1-z)} \tag{62a}$$

$$h = -Pr \tag{62b}$$

We mention that for any $\gamma + 1 = -k$ where $k = 0, 1, 2, \ldots$ Kummer's function $M(1+\gamma, 1+Pr, x)$ becomes proportional to the generalized Laguerre polynomial $L_k^{(Pr)}(x)$ and thus in all of these cases Θ given by Eq. (56) is of the form (a polynomial of z of degree k)$\times z^{Pr} e^{Pr(1-z)}$.

5.4. The case $\beta = -1$

In this case $Y = z - z^2/2$, $f_\infty = \sqrt{2}$ and $S = 0$ (see Eq. (42)). By the simple variable transformation

$$z = 2\xi \tag{63}$$

the energy equation (46) goes over in

$$\xi(1-\xi)\frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2}+\bigl(1-Pr-2(1-Pr)\xi\bigr)\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}-2\gamma\, Pr\,\Theta=0 \tag{64}$$

which is precisely the differential equation of Gauss' hypergeometric functions $F(a,b;c;\xi)$ with

$$ab=2\gamma Pr \tag{65a}$$

$$a+b=1-2Pr \tag{65b}$$

$$c=1-Pr \tag{65c}$$

(see Abramowitz and Stegun [22]). Thus, the solution which satisfies the boundary conditions (47) can be put in the form

$$\Theta=\frac{1}{2}(2Y)^{Pr}\frac{F(1-a,1-b;1+Pr;z/2)}{F(1-a,1-b;1+Pr;1/2)} \tag{66}$$

The heat transfer coefficient (48) becomes in this case

$$h=\frac{\sqrt{2}}{2}\frac{(1+\gamma)Pr}{1+Pr}\frac{F(2-a,2-b;2+Pr;1/2)}{F(1-a,1-b;1+Pr;1/2)} \tag{67}$$

Having in mind that in this case, according to Eqs. (42) and (44a), $z=1-\tanh(\eta/\sqrt{2})$, we recover in (66) and (67) the results obtained by Magyari and Keller [24] by standard calculations (see also [6,10]).

In order to be more specific we list below a couple of special cases of Eqs. (66) and (67).

The special case $\gamma=-1$. In this case $a=1$, $b=-2Pr$ and thus

$$\Theta=\frac{1}{2}(2Y)^{Pr}=\frac{1}{2}\left(1-\frac{f^2}{2}\right)^{Pr} \tag{68a}$$

$$h=0 \tag{68b}$$

The special case $\gamma=0, Pr=1/2$. Taking into account that

$$F\left(1,1;\frac{3}{2};x^2\right)=\frac{\arcsin x}{x\sqrt{1-x^2}} \tag{69}$$

(see Abramowitz and Stegun [22]) we obtain in this special case

$$\Theta=\frac{2}{\pi}\arcsin\sqrt{\frac{z}{2}} \tag{70a}$$

$$h=\frac{\sqrt{2}}{\pi} \tag{70b}$$

5.5. The general case

The above results suggest that the factor z^{Pr} could represent a general ingredient of the solution of energy equation. Moreover, $\Theta=z^{Pr}/f_\infty^2$ satisfies the boundary conditions $\Theta|_{z=0}=0$, $\Theta|_{z=1}=f_\infty^{-2}$ exactly and, as $z\to 0$ it also satisfies the energy equation approximately. Thus, it is reasonable to search for the solution of energy equation in the form

$$\Theta=z^{Pr}G(z) \tag{71}$$

where $G(z)$ is an unknown function. Substituting (71) in the energy equation (46) we obtain for G the following equation

$$\begin{aligned}&z^2YG''+z^2Y'G'+2Pr\,zYG'+Pr\,zY'G\\&\quad+Pr(Pr-1)YG+(z-1)(Pr^2zG+Pr\,z^2G')\\&\quad-\gamma Pr\,z^2G=0\end{aligned} \tag{72}$$

Here the primes denote derivatives with respect to z. Now we are looking for a series solution of Eq. (72) of the form

$$G(z)=K\sum_{k=0}^{\infty}B_kz^k \tag{73}$$

where K is a constant and $B_0=1$. In this way, having in mind Eq. (29), we obtain for the next two coefficients

$$B_1=-\frac{Pr}{1+Pr}\bigl[(1+Pr)A_2+Pr-\gamma\bigr], \tag{74a}$$

$$B_2=-\frac{A_3}{2}Pr-\left[(1+Pr)A_2+\frac{Pr(1+Pr-\gamma)}{2+Pr}\right]\frac{B_1}{2} \tag{74b}$$

The subsequent coefficients $B_3, B_4, B_5, \ldots$ can then be calculated recursively according to

$$\begin{aligned}B_k=&-\frac{1}{k(Pr+k)}\bigl\{Pr(Pr+k)A_{k+1}+Pr(Pr+1)A_kB_1\\&+\bigl[2(Pr+k-1)A_2+Pr(Pr+k-1-\gamma)\bigr]B_{k-1}\bigr\}\\&-\frac{1}{k(Pr+k)}\sum_{n=2}^{k-1}\bigl\{(Pr+k-n)\\&\times\bigl[(Pr+k-n+1)A_nB_{k-n+1}\\&+(n+1)A_{n+1}B_{k-n}\bigr]\bigr\},\quad k=3,4,5,\ldots\end{aligned} \tag{75}$$

Having in mind the boundary condition (47b) we obtain for the constant K the value

$$K=\left[f_\infty^2\sum_{k=0}^{\infty}B_k\right]^{-1} \tag{76}$$

Therefore

$$\theta=f_\infty^2\Theta=\frac{z^{Pr}\sum_{k=0}^{\infty}B_kz^k}{\sum_{k=0}^{\infty}B_k} \tag{77}$$

The heat transfer coefficient (48) becomes in this way

$$\begin{aligned}h&=\frac{Pr+\sum_{k=1}^{\infty}(k+Pr)B_k}{f_\infty\sum_{k=0}^{\infty}B_k}\\&=\frac{(Pr+\sum_{k=1}^{\infty}(k+Pr)B_k)(\sum_{k=1}^{\infty}A_k)^{1/2}}{\sum_{j=0}^{\infty}B_k}\end{aligned} \tag{78}$$

As an illustration in Fig. 4 the heat transfer coefficient h given by Eq. (78) is plotted as a function of the Prandtl number for the uniformly moving surface, $\beta=0$, and three different values of the temperature exponent, $\gamma=-0.5, 0$ and 1, respectively. In Fig. 5 the dimensionless temperature profiles $\theta=f_\infty^2\Theta(z)$ given by Eq. (77) are plotted as functions of the dimensionless stream function f, $0\leqslant f\leqslant f_\infty$ for $\beta=\gamma=0$ (uniformly moving surface of constant temperature) and three different values of the Prandtl number, $Pr=0.5, 1$ and 3, respectively. In all these cases, $S=-0.62758$ and $f_\infty=1.1427$.

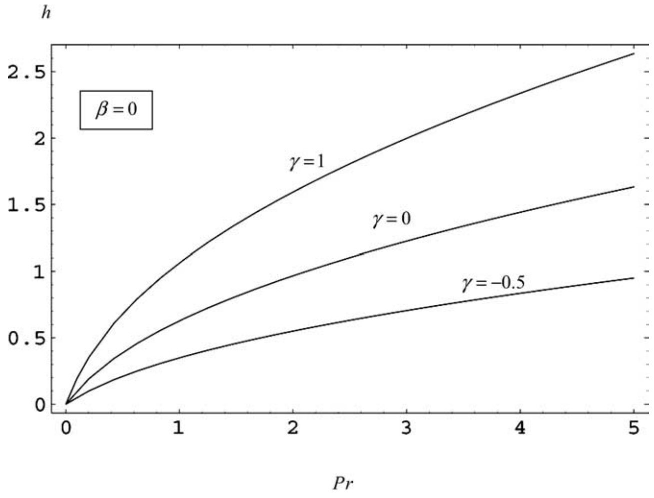

Fig. 4. Plot of the heat transfer coefficient h given By Eq. (78) as a function of the Prandtl number Pr for the uniformly moving surface, $\beta = 0$, and three different values of the temperature exponent γ, $\gamma = -0.5, 0$ and 1, respectively.

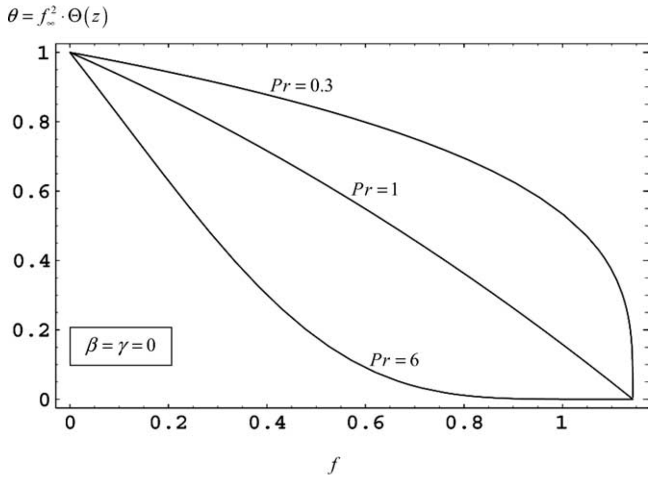

Fig. 5. Plot of the dimensionless temperature profiles $\theta = f_\infty^2 \Theta(z)$ given by Eq. (77) as functions of the dimensionless stream function f, $0 \leqslant f \leqslant f_\infty$ for $\beta = \gamma = 0$ (uniformly moving surface of constant temperature) and three different values of the Prandtl number, $Pr = 0.3$, 1 and 6, respectively. In all these cases, $S = -0.62758$ and $f_\infty = 1.1427$.

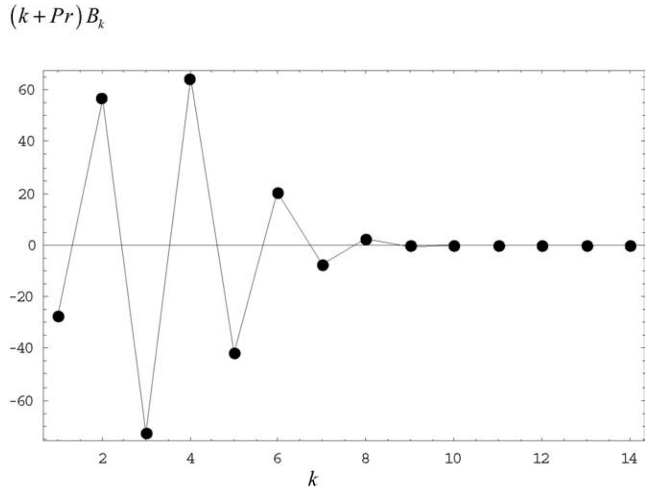

Fig. 6. The values of $(k+Pr)B_k$ as plotted for an isothermal ($\gamma = 0$) surface which moves uniformly ($\beta = 0$) in water ($Pr = 6.21$ at 25 °C). Behind the term $k = 14$ the values of $|(k + Pr)B_k|$ become smaller than 10^{-6}. The values of the coefficients $|A_k|$ do so already behind the term $k = 8$.

6. Discussion and conclusions

The "direct method" presented in this paper allows to calculate the main mechanical and thermal characteristics (skin friction, entrainment velocity and heat transfer coefficient) of two large classes of boundary layer flows (flows induced by stretching surfaces and free convection flows in saturated porous media) without any knowledge of the solutions of the corresponding boundary value problems. This approach uses the slightly modified form, $z = 1 - f/f_\infty$ of the Merkin transformation [2] which reverses the role of the dimensionless stream function f from that of a dependent to one of an independent variable (and thus it is closely related to the classical von Mises transformation [1]). The new dependent variables, which are the dimensionless velocity $Y = f'/f_\infty^2$ and the dimensionless temperature θ are obtained as power series of z according to Eqs. (29) and (77), respectively. The coefficients A_n and B_n of these series expansions, result as solutions of systems of algebraic recurrence equations, (31)–(33) and (74), (75), respectively, which are suitable for a rapid numerical evaluation of the coefficients to any order, in every specified case of interest. The coefficients A_n depend only on the stretching parameter β, Eq. (20a), and the coefficients B_n on the temperature parameter γ, Eq. (20b), on the A_n's and on the Prandtl number. Finally, the dimensionless entrainment velocity f_∞, the skin friction S and the heat transfer coefficient h are obtained in an explicit analytic form from Eqs. (35), (36)–(38) and (78), respectively.

Basically, the quantities f_∞, S and h can be calculated in general to any desired precision. However, in order to do this, it is important to have exact information about the convergence of the series involved. Owing to the easy solvability of the recurrence equations which yield the coefficients A_n and B_n, the information required about the convergence can be gained in a simple practical way without the use of abstract convergence criteria. Indeed, the numerical values of any number of coefficients A_n and B_n can be calculated and tabulated or represented graphically for any given values of the parameters β, y and Pr by using simple library programs. As an illustration in Fig. 6 the values of $(k + Pr)B_k$ (which are necessary to calculate the value of h according to Eq. (78)) are plotted for the practically important case of an isothermal ($\gamma = 0$) surface which moves uniformly ($\beta = 0$) in water ($Pr = 6.21$ at 25 °C). The values of $|(k+Pr)B_k|$ increase from $|-27.37|$ for $k = 1$ to $|-72.51|$ for $k = 3$ and then decrease with increasing k subsequently such that, e.g., for $k \geqslant 14$ they become smaller than 10^{-6}. In this time the values of the coefficients $|A_k|$ decrease subsequently from 1 for $k = 1$ such that for $k \geqslant 8$ they also become smaller than 10^{-6}. Similarly, for $\beta = +1$, $\gamma = 0$ and $Pr = 1$, e.g., the value of h obtained from Eq. (78) with the first 15 terms of the series involved, reproduces the exact result (60b) with an accuracy of 15 digits.

References

[1] R. von Mises, Bemerkungen zur Hydrodynamik, J. Appl. Math. Mech. 7 (1927) 425–431.
[2] J.H. Merkin, A note on the solution of a differential equation arising in boundary-layer theory, J. Engrg. Math. 18 (1984) 31–36.
[3] E. Magyari, Direct linear stability analysis for solitary waves, Phys. Rev. A 31 (1985) 1174–1176.
[4] B.C. Sakiadis, Boundary-layer behaviour on continuous solid surfaces, AIChE J. 7 (1961) 26–28; 221–225; 467–472.
[5] W.H.H. Banks, Similarity solutions of the boundary layer equations for a stretching wall, J. Méc. Theor. Appl. 2 (1983) 375–392.
[6] N. Afzal, Heat transfer from a stretching surface, Internat. J. Heat Mass Transfer 36 (1993) 1128–1131.
[7] E. Magyari, M.E. Ali, B. Keller, Heat and mass transfer characteristics of the self-similar boundary-layer flows induced by continuous surfaces stretched with rapidly decreasing velocities, Heat Mass Transfer 38 (2001) 65–74.
[8] M.E. Ali, The buoyancy effects on the boundary layers induced by continuous surfaces stretched with rapidly decreasing velocities, Heat Mass Transfer 40 (2004) 285–291.
[9] E. Magyari, B. Keller, Exact solutions for the self-similar boundary-layer flows induced by permeable stretching walls, European J. Mech. B Fluids 19 (2000) 109–122.
[10] E. Magyari, B. Keller, Reynolds' analogy for the thermal convection driven by nonisothermal stretching surfaces, Heat Mass Transfer 36 (2000) 393–399.
[11] P. Cheng, W.J. Minkowycz, Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike, J. Geophys. Res. 82 (1977) 2040–2044.
[12] I. Pop, D.B. Ingham, Convective Heat Transfer: Mathematical and Computational Modeling of Viscous Fluids and Porous Media, Pergamon, Oxford, 2001.
[13] S. Goldstein, On backward boundary layers and flow in a convergent passage, J. Fluid Mech. 21 (1965) 33–45.
[14] H.K. Kuiken, A backward free-convective boundary layer, Q. J. Mech. Appl. Math. 34 (1981) 397–413.
[15] E. Magyari, B. Keller, Backward free convection boundary layers in porous media, Transport Porous Media 55 (2004) 285–300.
[16] H. Schlichting, K. Gersten, Grenzschicht-Theorie, Springer, Berlin, 1997.
[17] E. Magyari, I. Pop, B. Keller, The "missing" similarity boundary layer flow over a moving plane surface, J. Appl. Math. Phys. 53 (2002) 782–793.
[18] D.B. Ingham, S.N. Brown, Flow past a suddenly heated vertical plate in a porous medium, Proc. Roy. Soc. London A 403 (1986) 51–80.
[19] L.J. Crane, Flow past a stretching plate, J. Appl. Math. Phys. 21 (1970) 645–647.
[20] W.B. Bickley, The plane jet, Philos. Mag. 23 (1937) 727–731.
[21] H. Schlichting, Laminare Strahlausbreitung, J. Appl. Math. Phys. 13 (1933) 260–263.
[22] M. Abramowitz, Handbook of Mathematical Functions, Dover, New York, 1965.
[23] L.J. Grubka, K.M. Bobba, Heat transfer characteristics of a continuous stretching surface with variable temperature, ASME J. Heat Transfer 107 (1985) 248–250.
[24] E. Magyari, B. Keller, Heat transfer characteristics of the separation boundary layer flow induced by a continuous stretching surface, J. Phys. D: Appl. Phys. 32 (1999) 2876–2881.